

# The continuous time random walk, still trendy: fifty-year history, state of art and outlook<sup>\*</sup>

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**Abstract.** In this article we demonstrate the very inspiring role of the continuous-time random walk (CTRW) formalism, the numerous modifications permitted by its flexibility, its various applications, and the promising perspectives in the various fields of knowledge. A short review of significant achievements and possibilities is given. However, this review is still far from completeness. We focused on a pivotal role of CTRWs mainly in anomalous stochastic processes discovered in physics and beyond. This article plays the role of an extended announcement of the Eur. Phys. J. B Special Issue [<http://epjb.epj.org/open-calls-for-papers/123-epj-b/1090-ctrw-50-years-on>] containing articles which show incredible possibilities of the CTRWs.

## 1 Inspiring properties and achievements

In their pioneering work published in year 1965 [1], physicists Elliott W. Montroll and George H. Weiss introduced the concept of continuous-time random walk (CTRW) as a way to make the interevent-time continuous and fluctuating. It is characterized by some distribution associated with a stochastic process, giving an insight into the process activity. This distribution, called pausing- or waiting-time one (WTD), permitted the description of both Debye (exponential) and, what is most significant, non-Debye (slowly-decaying) relaxations as well as normal and anomalous transport and diffusion [2,3] – thus the model involves fundamental aspects of the stochastic world – a real, complex world. Notably, ancestors of this concept are presented by Michael Shlesinger in [4].

Let us incidentally comment that term “walk” in the name “continuous-time random walk” is commonly used in the generic sense comprising two concepts: namely, both the walk (associated with finite displacement velocity of the process) and flight (associated with an instantaneous displacement of the process). Thus we have to specify in a detailed way with what kind of process we are considering.

The CTRW formalism was most conveniently developed by physicists Scher and Lax in terms of recursion relations [5–10]. In this context the distinction between

discrete and continuous times [11] and also between separable and non-separable WTDs were introduced [12]. A thorough analysis of the latter, called also the nonindependent CTRW, was performed a decade ago [13] although, in the context of the concentrated lattice gases, it was performed much earlier [14,15]. These analyses took into account dependences over many correlated consecutive particle displacements and waiting (or interevent) times.

The Scher and Lax formulation of the CTRW formalism is particularly convenient to study as well anomalous transport and diffusion as the non-Debye relaxation and their anomalous scaling properties (e.g., the nonlinear time growth of the process variance). Examples are the span of the walk, the first-passage times, survival probabilities, the number of distinct sites visited and, of course, mean and mean-square displacement if they exist. It is very interesting that all these things are also used to characterize complex systems [16].

In principle, the CTRW is fundamentally different from the regular random flight or walk models as the probability density of the flight or walk in the long-time (asymptotic) limit scales in a non-Gaussian way [17]. Thus, the CTRW became a foundation of anomalous (dispersive, non-Gaussian) transport and diffusion [10,18,19]. This opened the modern and trendy segment of statistical physics, as well as condensed and soft matter physics, stimulating their very rapid expansion outside the traditional (Boltzmann-Gibbs) statistical physics (including statistical physics of open systems [20]) [21,22].

The variety of observed relaxation phenomena in condensed and soft matters are related to transport and/or

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diffusion most often of atoms, particles, carriers, defects, excitons, and complexes [23]. Transport and diffusion are regarded, in fact, as a paradigm of irreversible behaviour of many ordered and disordered systems. The fundamental significance of irreversible behaviour arising from coarse graining Hamiltonian systems with the help of CTRW-like waiting time distributions was emphasized in [24–27]. A universal feature of disordered systems is a temporal complex pattern, which the Debye relaxation no longer obeys.

The carrier transport in some amorphous insulators (such as the commercially used vitreous  $\text{As}_2\text{Se}_3$ ) and in some amorphous charge-transfer complexes of organic polymers (as the commercially used trinitrofluorenone mixed with polyvinylcarbazole, TNF-PVK) provides canonical examples of (i) continuous-time random flights and walks and (ii) broad- or long-tailed WTDs. The generic description of the dispersive transport and diffusion [28] found in the breakthrough experiments on transient current in an amorphous medium, induced by flash light [18,29–32] or voltage pulse [33], is actually given by the continuous-time random walk formalism where carrier displacements have an instantaneous character; that is, carriers perform flights instead of walks.

Although originally the CTRW was a kind of renewal theory, Tunaley was able to modify it by preparing the class of initial (averaging) WTD. Such a modification makes time homogeneous [34–37] and enables us to consider CTRW as a semi-Markov process [38,39]. Thus the application of the key Wiener-Khinchin (WK) theorem (relating autocorrelation function to power spectra) became possible. In other words, there are two categories of initial conditions: the first one, called equilibrated or stationary, wherein, before starting observation of the process, the system is allowed to equilibrate [40], and the second category, called non-equilibrated or non-stationary, where evolution starts at a given instant without any knowledge of the past [41]. Recently, Leibovich and Barkai have generalized WK theorem to the widely observed non-stationary processes, which describes the power spectrum of CTRWs (when the average sojourn time diverges). Thus, connection between the power spectrum and the scale invariant correlation function is obtained. Hence, for instance, the power spectrum of blinking quantum dots is described as a two-state CTRW process. This topic is really hot out of the oven; two experiments have already been performed on this topic. Since this subject is related to  $1/f$  noise, it still has aroused widespread interest. By using CTRW formalism one can focus on the origin of  $1/f$  noise, something that was known for a while [42,43] (see also some remarks below).

There is another extremely significant aspect of the above-defined problem of initial conditions, namely the dependence of the formalism upon them, which leads directly to ergodicity breaking in the Boltzmann sense. Note that the ergodicity in the Boltzmann sense means that, for a sufficiently long time, the time average tends toward the ensemble average. Ergodicity breaking has been related to stationarity breaking, irreversibility and coarse grained

dynamics on subsets of measure zero in [44–46]. One can say that there are two essential interrelated problems of the CTRW: (i) the initial preparation of the system, important for all random walk models, and (ii) the weak ergodicity breaking, which is of great interest both from theoretical and empirical points of view.

The diverging of a microscale (or average waiting time diverging) for the non-Debye relaxation and for anomalous transport and diffusion leads to the property of weak ergodicity breaking [47–56]. Although then the whole phase space of the system can be still explored, the ergodicity is never obeyed because any measurement time is always shorter or of the order of the time characteristic for the process considered. This property was introduced in the context of processes showing aging, found experimentally, for instance, in the diffusion of ion channels on the membrane of living cells [55] and in blinking quantum dots [57].

In order to say something deeper about the non-ergodic properties of the CTRW processes, we should first note that anomalous diffusion loses the universality of Brownian motion; that is, the mean-square displacement is no longer sufficient to uniquely identify a stochastic process. Therefore, various stochastic processes give rise to anomalous diffusion exhibiting many different features. For instance, for subdiffusive renewal CTRWs the ergodicity is violated, and even weak ergodicity breaking was observed due to the diverging characteristic waiting time [58]. Ergodicity breaking effects are essential in understanding fluctuation-generated phenomena, in particular fluctuation dissipation relations and linear response [59].

Notably, modern empirical single particle tracking techniques and many large scale simulations producing time series of the position of a tracer particle fully confirmed this observation [60]. A non-ergodic generalization of the Boltzmann-Gibbs statistical mechanics for systems with infinite mean sojourn time was found [48], which seems to be a great achievement.

It was shown by using extreme value theory (EVT), that indeed rare or extreme events actually govern dispersive transport and diffusion [61,62]. Thus the CTRW formalism opened promising opportunities to connect a microscopic stochastic dynamics of objects to the macroscopic processes of relaxation, transport and diffusion.

The biased CTRW formalisms were also developed, wherein the bias can affect both spatial and temporal variables [63,64]. Such an approach constituted a basis for the origin of  $1/f$  noise – attributed to the excess noise following from the process fluctuation.

The appearance of  $1/f$  noise can be achieved, for instance, by appropriately modeling trapping states and determining the correspondence waiting-time distribution. The  $1/f$  noise can affect several thermally activated processes; e.g. it can govern anomalous escape [65].

The biased and non-biased aging continuous-time random walks, using fractal renewal theory, were prepared [66]. Thus an essential extension of the canonical (non-stationary) CTRW formalism was developed which

contains the aging period prior to relaxing continuous-time random walk.

The canonical version of the CTRW formalism, concerning transitions between different sites and states considered recursion relations, is equivalent to the form of a generalized master equation (GME) as a one-to-one transformation between WTD and memory kernel was clearly established [67–76]. Originally, this was used to exhibit the equivalence between models of hopping and multi-trapping in amorphous materials. Notably, a kernel without memory corresponds to the ordinary master Markovian equation, while exponentially decaying memory enables us to transcribe GME to a variant of the telegrapher's equation [77] when the short-time mechanism of the random walk is activated.

The non-Markovian nature of the CTRW is seen, in particular, in the presence of disorder in the system, considered in the ensemble average. The term 'non-Markovian' means that the current state of the particle depends on all its history, beyond the recent one. An equivalence between averaged particle transport in disordered systems and the GME or CTRW formalism was established [78]. Before averaging, the individual disordered systems are Markovian. However, when master equations are averaged over the disorder, by using Zwanzig-Nakajima projection formalism [79,80], it leads to the GME that is, the memory kernel appears as a result of a double average over possible random walk trajectories and over imperfections present in the system. Generally speaking, the vast majority of versions of CTRW formalisms (not only those referring to disordered systems) can be treated as renewal semi-Markov processes.

Exact results for the extreme value statistics of CTRW was obtained for the first time by using the real-space or strong disorder renormalization-group (RG) [81]. Moreover, the RG treatment of the canonical CTRW formalism was performed by using a decimation procedure in one dimension [82,83]. The resulting fixed-point equation for WTD gave an excellent analytical solution which does not correspond to the Poisson process in a macroscale (although initially, in a microscale, it did so) and, paradoxically, does not depend upon whether disorder is present in the system or not. This shows that the CTRW formalism is suited to study systems near criticality, insensitive on model details.

The correlated CTRWs belong to a class of RW where memory is not lost after each step. They were found in various physical applications, e.g., in conformation of polymers [84,85], tracer diffusion in metals [86,87] and mixed-alkali superionic conductors [88]. The simplest version of these CTRWs assumes correlation over two successive steps, both persistent and antipersistent. Such a model is equivalent with a second-order integro-differential equation [89,90]. Notably, in several papers the properties of persistent and antipersistent CTRWs were considered [91,92], e.g., in the context of determination of Hurst and Hölder exponents. Therefore, it seems a natural way to adopt CTRW to study multifractal types of random walk. This can be based on the well-known

characterization of the stochastic phenomena using a spectrum of fractional moments, both temporal and spatial [28,93–101]. The scaling of the fractional moments of the Lévy walk has a characteristic fractal, bifractal, and multifractal behaviour. Besides, the common observation was that the diffusion coefficient and conductivity are then frequency dependent quantities which are similar to that observed experimentally, e.g., for ionic conductors. Moreover, the persistent CTRW leads to frequent use of the telegrapher's equation [12].

A model of subdiffusion, where waiting times are mutually dependent (i.e. intrinsically correlated), was also developed by using stochastic dynamic (Langevin) equations. This led to anomalous diffusion under the influence of an external force field [102]. A very helpful effective theoretical criterion for subdiffusion was found in the context of the reference fractional Brownian motion [103]. It is highly characteristic that subdiffusion has been found in very different systems, beginning with the seminal discovery of charge carrier anomalous transport in amorphous (glossy) semiconducting films mentioned above (in this context the CTRW formalism in the presence of traps was also developed [104]). Other attractive examples could be tracer dispersion in subsurface aquifers [105,106], hiking on percolating clusters [107] or even bacteria in biofilms [108] and tracers in crowded media such as living biological cells [109–112].

The CTRW formalism is well suited to study first-passage time (FPT) problems related to trap and escape problems, using the so-called survival probability (SP) directly expressed by the WTD [12]. Numerous physical applications of these problems exist. For instance, FPT of a randomly accelerated particle, FPT problems in anomalous diffusion, FPT of intermittent random walks, FPT phenomena on finite inhomogeneous networks, FPT of network synchronization, FPT statistics for random walks in bounded domains, FPT behavior of multi-dimensional fractional Brownian motion and application to reaction phenomena. A persistence and first-passage properties in non-equilibrium systems are of great importance not only in this context [113].

Exact relations for the path probability densities of a broad class of anomalous diffusion processes were derived by employing path integral formulation [114]. A closed analytical solution for the path probability distribution of a continuous time random walk process was derived. This analytical solution is given in terms of the corresponding waiting time distribution and short time propagator coming from the Dyson equation. For instance, by applying this analytical solution, the generalized Feynman-Kac formula was found.

A random walk along the backbone over comb with teeth lengths varying according to power-law distribution, is another interesting example of the CTRW application [115]. As a result, the CTRW predicts anomalous diffusion (subdiffusion) when local WTD (i.e. conditional at given teeth length) is also given by power-law. Anomalous diffusion in comb-like structures serves as a reference model for random walk in more complicated fractal

substrates such as percolation clusters successfully reproducing the delaying effects of dangling ends and the backbone irregularities in percolation clusters.

Thanks to its versatility, the CTRW found numerous important applications in many fields ranging from biology through telecommunication to finance including econometrics and economics, and even to speech recognition. The CTRW found innumerable applications in many other fields, still growing, such as the aging of glasses, a nearly constant dielectric loss response in structurally disordered ionic conductors and in modeling of hydrological models and earthquakes. Since the canonical CTRW was first successfully applied by Scher and Lax in 1973 [6] (and independently by Moore one year later [116]), to describe an anomalous transient photocurrent in an amorphous glassy material (manifesting the power-law or non-Debye relaxation), this formalism has achieved much more than its original goal. An approachable description of fractional kinetics with characteristic applications to anomalous charge transport and relaxation in solids such as, for example, disordered semiconductor structures, quantum dots and wires, dielectrics (polymers and ceramics), as well as nanosystems, can be found in reference [117].

## 2 CTRW formalisms vs. fractional evolution equations

Apparently, a modern era of the CTRW formalism, i.e., its “second youth” or “revival”, started with the discovery of the rigorous relation between the CTRWs and time-fractional diffusion equations (time-FDEs) made by Hilfer in 1995 [118,119] (see also Sect. 2.3.4.3 in [120]). The rigorous relation elaborated in [121–124], has become a fruitful beginning for subsequent investigations, particularly of fractional Fokker-Planck equations with drift (discussed in Sect. 2.8).

Time-fractional diffusion of Montroll and Weiss is (to our knowledge) the only known example of a fractional differential equation, where the order of the fractional derivative can be derived from an underlying microscopic model, wherein the microscopic model does not contradict the fundamental principle of locality in modern theoretical physics [120]. In contrast, the space-fractional Bochner-Levy-Riesz diffusion models based on a fractional Laplace operator (see Sects. 2.2, 2.3, 2.8, 2.11, and 2.13) have remained more speculative, because they predict non-local experimental phenomena (see Sect. 2.3.2 in [120] and also [125]).

The paper [126] of Mainardi, Raberto, Gorenflo, and Scalas is impressive in this context. They derived (in the frame of the CTRW) an alternative version of the GME and hence the time-fractional Kolmogorov-Feller type equation (see also Sect. 2.6), considering long-term weakly singular memory that exhibits power-law time decay. In this equation, the partial time derivative was naturally replaced by the Caputo fractional derivative, without use of the Riemann-Liouville fractional one [127]. It should be noted that this form of memory kernel implies,

e.g., sojourn probability in the form of the Mittag-Leffler function and the corresponding WTD as its ordinary derivative. It is a promising extension as these quantities also appear in fractional relaxation and fractional oscillation processes [128–130]. Notably, the sojourn probability reproduced the dynamics of BUND future prices traded in LIFFE quite well, with various delivery dates in 1997 [126]. In sum, the approach mentioned above seems to be very useful for financial analysis.

Evidence appeared (see Sects. 2.1–2.13 below) that various models of fractional stochastic dynamics and kinetics, as well as fractional stochastic processes, are in macroscopic limit (that is, for the asymptotic long time and for large values of space variable) equivalent to the CTRW formalism. It should be kept in mind, however, that the power law tails do not always lead to FDEs. The importance of appropriate scaling limits in this context has been emphasized in [27,124].

### 2.1 Fractional diffusion equation

In three different ways the equation of diffusion can be modified to fractional diffusion equation (FDE) so as to be equivalent to the asymptotic form of the canonical CTRW formalism, i.e. the separable (independent or decoupled) and uncorrelated continuous-time random walk formalism characterized by diverging mean waiting time reflecting the existence of deep traps in the system. This means that the average depth of the trap is greater in this case than the Boltzmann constant multiplied by the absolute temperature.

The first way leads to FDE describing subdiffusion, where the partial-time derivative is replaced by the corresponding Marchaud fractional derivative [121]. The second way leads to FDE where the Laplacian in the diffusion equation is completed with the partial time differentiation and with the Riemann-Liouville fractional differentiation [28,131]. In this way even more general FDE is reached where Laplacian is replaced by the Riesz-Weyl fractional derivative [28]. Thanks to this, the competition between long-lived rests (waiting) of the particle and its long jumps (flights) produces a rich phase diagram consisting of four different random walk phases: canonical Lévy flight (LF), non-Markovian LF, Brownian diffusion, and subdiffusion. Notably, the fractional diffusion-advection equation (FDAE), containing the usual advection term, was easily obtained in this way within the laboratory frame of reference. In this case, Galilei invariance is obeyed between laboratory and moving frame of references, where the one-dimensional velocity field is homogeneous. Otherwise, when the velocity field is inhomogeneous, the resulting FDAE has the same structure as the fractional Fokker-Planck equation (discussed in the next paragraph).

### 2.2 Fractional kinetic equation

The third, hybrid way, which combines the approaches mentioned above, leads to the generic FDE with a long-term source containing the initial (static) distribution,

wherein both time and space fractional Riesz/Weyl derivatives were used [3,28,132,133]. This equation (also called the fractional kinetic equation (FKE)) has several interesting regimes covering, besides the case of normal diffusion, also sub- and superdiffusion – the latter considered also in Section 2.3 by Lévy fractional diffusion equation.

Notably, solutions of the FDEs presented above were obtained within closed analytical forms, but only the second and third ways give the canonical Lévy distribution as a solution (for proper combination of scaling exponents) [3].

### 2.3 Lévy fractional diffusion equation

The Lévy flights (LFs) belong to the class of Markov process with broad flight-length distribution, possessing asymptotically the inverse power-law behaviour such that its variance diverges. This scale-free ‘area-unfilling’ or fractal nature of this distribution gives, in force-free cases, the Lévy fractional diffusion equation (LFDE) [132,134–136]. In this diffusion equation, the fractional Riesz/Weyl derivative replaced the Laplacian, and fractional diffusion coefficient appeared instead of the usual diffusion coefficient. The time-dependent solution of the LFDE has a closed form which can be represented by the Fox  $H$ -functions [28,137]. Obviously, this solution gives well-known Lévy stable law.

Due to the Markovian character of the LFDE, the constant drift velocity can be easily incorporated into the LFDE in the form of a usual drift or advection term [28,138]. This is possible if mean waiting time is finite. This equation can be called the Lévy fractional diffusion-advection equation (LFDAE). The closed solution of this extended equation is given, in fact, by that for the free case with the space variable shifted by displacement caused by the drift.

### 2.4 Distributed-order fractional diffusion equation

Anomalous non-scaling behavior, corresponding either to a non-power-law (e.g., the logarithmic) growth of a distribution width or to a crossover between different power laws, are observed quite often. Commonly known examples of such a behavior are the Sinai-like superslow subdiffusion and superdiffusive truncated Lévy flights. All of them can be described by diffusion equations with distributed-order (fractional) derivatives [139]. We are dealing with a distributed-order derivative when average over orders of its corresponding fractional components is made. Two generic cases should be clearly distinguished herein: (i) the distributed-order time fractional diffusion equation and (ii) the distributed-order space fractional diffusion equation. For both cases (implemented through their different forms) weight plays a central role, and relations to the CTRW are well established.

### 2.5 Fractional telegrapher’s equation

In the context of diffusion theory, the telegrapher’s equation (TE) is seen as a relativistic generalization of the diffusion equation, since the latter is not compatible with relativity [140–144]. It also takes into account ballistic motion, and tends to be more accurate in modeling transport near boundaries than the diffusion equation [145]. Within the surge of anomalous diffusion during the last two decades, there have been some (few) attempts to generalize TE to include fractional motion. Thus, in the mathematics literature, there have been some works analyzing mathematical and other formal properties of a fractional version of the TE. However, the fractional equation is set in an ad hoc fashion, just by replacing ordinary derivatives by fractional derivatives that can be of various types [146–148]. Efforts meant to derive the fractional telegrapher’s equation (FTE) based on physical grounds are very scarce [149–151]. A rather thorough derivation of the one-dimensional FTE [152] based on the (fractional) persistent random walk has been presented very recently. This is a variant of the CTRW which allows for the presence of internal states and incorporates a form of momentum within the framework of diffusion theory [145,153].

### 2.6 Fractional Kolmogorov-Feller equation

The well-known Kolmogorov-Feller equation is the master equation used in different physical applications (see the first paragraph of this section). However, further generalization is required in order to describe the non-Markovian kinetics [132]. This was achieved by (i) replacing partial time derivative in the (usual) Kolmogorov-Feller equation by Riesz/Weyl, and (ii) extending this equation by the source term localized at the origin and slowly decaying in time.

### 2.7 Fractional master equation

There is also a fractional generalization of the conventional master equation for the subdiffusive random walk (initially located at origin), where the partial time derivative is replaced by the fractional one [118]. This generalization becomes the canonical CTRW formalism in the macroscopic limit by assuming the WTD which asymptotically exhibits a power-law tail.

### 2.8 Fractional Fokker-Planck equation

A framework for the treatment of anomalous diffusion problems under the influence of an external force field is presented here. This is a response to the fact that many transport and diffusion problems in science and technology take place under the influence of an external force field [28], sometimes near the thermal equilibrium [154].

An effort has been made by many authors to find the most general circumstances defining the above mentioned framework. A particularly inspiring, in some sense

more general than the FDE, supplies a one-dimensional nonhomogeneous fractional kinetic equation or fractional Fokker-Planck-Kolmogorov equation (FFPKE) describing a symmetrized in space wandering with a pointwise algebraically relaxing source [132], where both time and space derivatives are fractional.

In order to describe anomalous transport in the presence of an external field, the fractional Fokker-Planck equation was found, where the Fokker-Planck operator was completed with the partial time differentiation joint with the Riemann-Liouville fractional differentiation [28,131]. Its stationary solution is given (as for the usual FPE) by the Gibbs-Boltzmann distribution. For the force-free limit the FFPE reduces to the FDE mentioned in Section 2.1.

Taking also non-local jump statistics into account, i.e. assuming a jump-length distribution with infinite variance, one recovers generalized FFPE where Laplacian in the Fokker-Planck operator was replaced by the Riesz/Weyl fractional derivative. Thus it describes the competition between subdiffusion and Lévy flights. Obviously, if order of the Riesz/Weyl fractional derivative equals 2, the generalized FFPE reduces to the subdiffusive FFPE mentioned above. Notably, the FFPE enables one to study subdiffusion under the influence of time-dependent alternating force fields or driving [155,156].

Assuming the finite mean waiting time, the Markovian FFPE can be derived for Lévy flights, which is the usual FPE where only Laplacian is replaced by the Riesz/Weyl operator [28]. Such an FFPE describes systems far from thermal Boltzmann equilibrium. This is clearly seen in the behaviour of a particle underlying the harmonic potential, where the stationary state is defined by the Lévy stable law.

Moreover, the FFPE which contains a variable diffusion coefficient, is discussed and effectively solved [157]. It corresponds to Lévy flights in a nonhomogeneous medium. It is interesting that for the case with linear drift, the solution becomes stationary in the long-time limit representing the Lévy process with a simple scaling.

## 2.9 Generalized Langevin subdiffusion dynamics

There are two characteristic, essentially different mechanisms underlying the subdiffusive dynamics; that is, subdiffusion in the presence of a tilted washboard (nonlinear) potential energy profile [158]. The first approach is based on the fractional Fokker-Planck equation (derived within the continuous-time random walk), while the second approach is associated with the fractional Brownian motion in the form of a generalized Langevin equation (GLE) [159] containing memory-friction slowly relaxing term. Indeed, for such a potential the difference between both approaches becomes particularly distinct. For instance, it was found that the second approach is more ergodic (asymptotically ergodic) than the first one because the latter is based on the concept of fractal stochastic time with divergent mean period and finite mean residence time in a finite spatial domain. Moreover, the anomalous

transport coefficient became universal within the GLE, obeying the generalized Einstein relation (in the absence of periodic potential). Remarkably, the GLE subdiffusion is based on the long-range velocity–displacement correlation and not on diverging mean residence time in a potential well – the latter case is closely tightened to a weak ergodicity breaking. Hence, the contrast to the CTRW subdiffusion (with independent increments) clearly arises, showing that both approaches belong asymptotically to different universality classes. Concerning applications, the GLE subdiffusion dynamics seems to be appropriate, e.g., for regimes slightly above the glass transition or for crowded viscoelastic environments (like cytosols in biological cells).

## 2.10 Fractional Klein-Kramers equation: subdiffusion

The dynamics in phase space spanned by velocity and position, governed by the multiple trapping process (both subdiffusive and Markov limits) [28,131], is defined by the fractional Klein-Kramers equation (FKKE). The Klein-Kramers operator is in this equation completed with the partial time differentiation and with the Riemann-Liouville fractional differentiation. Note that the Stokes operator [160] is replaced in the FKKE by the corresponding fractional one, which shows the non-local drift response due to trapping [131,161–163].

One may consider the under- (velocity equilibration) and overdamped (large friction) limits. The former limit corresponds to the fractional version of the Rayleigh equation in the force-free limit [161,162,164]. This is a subdiffusive generalization of the Ornstein-Uhlenbeck process. In the overdamp case, the FKKE corresponds, in position space, to the FFPE [161,162,165,166]. In this case, the initial condition is persistent due to slow decay of the sticking probability. The generalized Einstein-Stokes relation and linear response in the presence of a constant field are obeyed.

## 2.11 Lévy fractional Klein-Kramers equation: superdiffusion

There are several ways to obtain the Lévy fractional Klein-Kramers equation concerning superdiffusion in phase space spanned by the velocity and position of a single flyer [163,167–170]. This equation is a generalized Klein-Kramers equation where the second-order partial derivative attached to the flyer velocity is replaced by the fractional Riesz/Weyl derivative. The Lévy FKKE allows the divergence of the flyer's kinetic energy [171] as a result of the linear (hydrodynamic) friction inherent in this equation being too small. Indeed this friction is a source of unphysical Lévy flights. These flights can be regularized by permission of the nonlinear (aerodynamic) friction, e.g., ballistically depending on the flyer velocity [131]. It should be added that the velocity average of the Lévy FKKE reduces it to the Lévy FFPE mentioned in Section 2.8.

There are also other ways to regularize the Lévy flights. For instance, particularly natural is the way which uses a non-separable CTRW, where hierarchical spatio-temporal coupling is exploited [172]. By term ‘hierarchical spatio-temporal coupling’ is understood a coupling between single-step displacement and preceding its waiting time separately on each level of the hierarchy of waiting-time distributions, extended over infinite many scales or levels. The finite mean-square displacement (MSD) was achieved then for arbitrary time thanks to the competition between flights and waitings – this competition produced a rich phase diagram (see Fig. 3 in Ref. [172] for details), where a (combined) diffusion exponent characterizes many diffusion phases defined by partial (i.e. spacial and temporal) scaling (or shape) exponents. Besides the normal diffusion phase, there are subdiffusion, enhanced diffusion, and ballistic diffusion phases. The latter phase defines border, separating these phases from the pure Lévy one (characterized by diverging MSD). It is worth noticing that even if extremely long flights are very likely (e.g. were drawn from Lévy stable law), the long waiting can compensate them after many steps, resulting, for instance, even in subdiffusive CTRW.

By the way, there are two characteristic models using spatio-temporal coupling, called the CTRW with “jump first” [173] and CTRW with “wait first” [174,175], which clearly show the influence of the first state on the particle dynamics. The former model assumes the particle jump as a first state while the latter assumes waiting instead of the jump. This is the only difference between these two models, which, however, leads to a distinct difference in the last step at given time  $t$ . This difference is crucial. Although propagators of both models have the same scaling properties, their shapes are model specific. It is worth mentioning that trajectories of the particle produced within these models resemble that of the standard Lévy walk model.

Moreover, the spatio-temporal coupling was similarly used in regularization of more complex processes, like Lévy walks [176] which have richer phase diagram (cf. Fig. 2 in this reference). It should be added that both Lévy flights and Lévy walks are observed in the real world [177].

## 2.12 Fractional Feynman-Kac equation

The fractional Feynman-Kac equation (FFKE) [178], although a topic not getting too much spot light, constitutes a convenient tool to study several characteristic functionals of the subdiffusive CTRW processes, both in the absence and presence of a binding force field (e.g. the harmonic field) [179]. In the latter case the route to weak ergodicity breaking was shown. The FFKE can be obtained from the usual (integer) Feynman-Kac (FK) equation by inserting to the FK one a substantial fractional derivative operator instead of the Laplacian operator and generalized diffusion coefficient instead of the usual one.

## 2.13 Lévy walks

When particle performs the CTRW of the flight type, then during its evolution it makes instantaneous jumps alternated with waiting events or rests. The CTRW formalism enables us to combine both particle states, offering an abundant diffusion phase diagram or several scaling regimes. Moreover, the CTRW formalism can be extended, assuming walks with finite fixed velocity instead of instantaneous jumps. Such a model is called the Lévy walk interrupted by rests [180–182]. Although the presence of finite particle velocity there significantly increases the flexibility of this kind of model, this simultaneously makes it more difficult to find their analytical solutions, if it exists. Obviously, the standard versions of Lévy walk model, i.e. without rests, were also intensively developed assuming fixed particle velocity [183,184] or varying, e.g. according to self-similar hierarchical structure [185].

There are several generalizations of the Lévy walk model which assume that particle velocity can vary randomly [186,187] or by some other rules [187]. Among them, Lévy walk models with random velocity, and particularly the one with weakly fluctuating velocity caused by the active environment [41,188,189], are very instructive and useful. In the frame of the former model, each displacement has its own velocity drawn from a given velocity distribution. Because of the additional complexity added through velocity distribution, few analytically solvable examples have been found. Among them the Lorentzian or Cauchy velocity distribution offers the prominent one. This velocity distribution appeared, for instance, in: (i) physical problems of two-dimensional turbulence [190–192]; (ii) as a model distribution of kinetic theory [193,194]; (iii) as a particular case of generalized kappa distributions of plasma physics applications [195,196]; and (iv) in some statistics [197,198]. Moreover, it was also found for the distribution of velocities of starving amoeba cells [199]. In the case of the latter model, particle velocity fluctuates around a fixed averaged value. As a result, the fluctuations accumulate with time and the final position of the particle, passing through the active medium, will differ from that produced by the standard Lévy walk.

Concluding this extremely important section, we can say that the finite velocity of walking particles constitutes random walk models, more general than the random flight (jump) ones, bringing them closer to physical principles and making them more suitable for description of real-life phenomena.

### 2.13.1 Useful tools and selected applications

One of the central question is how the Lévy walk emerges in diverse phenomena. Certain progress in this respect was achieved thanks to the development of necessary tools relevant for the Lévy walks. Among them the propagator of the single particle process and the space-time velocity autocorrelation function for such a process, together with its two-point generalization, are particularly useful.

Moreover, further extension of the space-time velocity correlation function to the broader class of initial conditions is also possible. It should be noted that all these functions perfectly characterize diffusion phases, both normal and anomalous. For many versions of the CTRW formalisms they can be calculated in analytically closed forms, at least asymptotically.

In the context of Lévy walks, the complementary quantities such as non-normalizable densities (caused by some singularities) are also exploited, making possible to calculate moment diverging within the standard Lévy distribution [99,100].

Anyway, the extension of Lévy walk models to higher dimensions is at the beginning stage of development as even extension to two dimensions encounters difficulties.

### 2.13.2 Chaotic advection, chaotic Hamiltonian dynamics, and Lévy walks with rest

Chaotic Hamiltonian systems, with their unique properties like cantori and stickiness phenomena, are extremely useful for two-dimensional chaotic advection [200]. Chaotic advection is the field at the intersection of fluid mechanics and nonlinear dynamics, which encompasses a range of multiscale applications ranging from micrometers to hundreds of kilometers, including systems as diverse as mixing and thermal processing of viscous fluids, micro-fluidics, biological flows, and large-scale dispersion of pollutants in oceanographic and atmospheric flows.

The two-dimensional velocity field of a passive scalar (tracer particle) within the incompressible flow can be expressed by the scalar stream function. That is, coupled equations describing the motion of the tracer particle (tracer dynamics) looks like the canonical Hamilton equations in classical mechanics with the stream function playing the role of the Hamiltonian, while both space coordinates of the passive scalar look like the conjugate coordinates. If the stream function (Hamiltonian) is time-dependent, the passive scalar (phase space point) trajectories can be chaotic. Indeed, this is chaotic advection which can occur even if the flow is laminar [201,202]. This was observed experimentally in a rapidly rotating annular tank where flow consists of an azimuthal chain of stable vortices sandwiched between two azimuthal inner and outer jets. If the flow has, e.g., periodic time dependence in the annulus reference frame (and can even be time-independent in the reference co-rotating with the vortex chain), the tracer typically follows chaotic trajectories alternately sticking near the vortices and occasionally walking ballistically in jet regions for long distances. The competition between sticking and walks can give rise to anomalous diffusion. Probability distribution functions are measured both for sticking and walk times resulting in power-laws. It was also discovered that the probability distribution function of azimuthal walk lengths is the Lévy one, and the long-term process as a whole can be considered as Lévy walk with rests. A very inspiring relation between Lévy walks with tracer-particle rests and strange kinetics was discovered in 1993 year [203]. Meanwhile, a real experiment of

Solomon-Weeks-Swinye [201,202] confirmed it. The term ‘strange kinetics’ refers to a kinetic description of a dynamical system exhibiting chaotic behaviour, where non-linearity in the Hamiltonian can induce fractal motions with exotic statistical properties.

### 2.13.3 Applications in optics

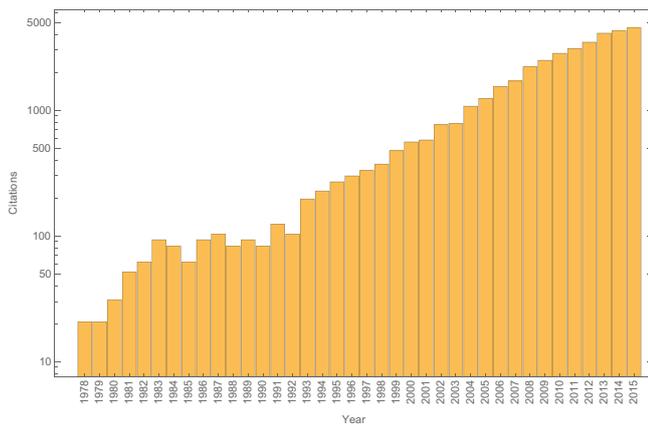
We present herein chosen exciting applications of Lévy processes (flights or walks) to model scattering of light by media. This is, in fact, a multiple scattering by medium inhomogeneities ruled by very different scattering mechanisms depending on the characteristic size and structure of inhomogeneities [3] (and references therein). For instance, the path of a photon inside a fractal medium can be represented by a random walk trajectory consisting of undisturbed segments connecting subsequent scatterers – the statistics of segments’ lengths is a power-law [204]. As photons move with finite velocity in any medium, the Lévy walk model is more appropriate to describe the photon dynamics than the Lévy flight, although, in some experiments, e.g. type of light transmission in the Lévy glass, the latter process gives a sufficient description.

The blinking quantum dots (QDs) is another modern example which can be mapped to the Lévy walk in the ballistic regime [205,206]. It is speculated that the Lévy walk model with random velocities could be useful for the interpretation of experiments with a whole distribution of intensities. Nevertheless, the microscopic mechanism responsible for the appearance of the power-law distributed blinking times in quantum dots remains unknown. Therefore, a real experiment of Krapf is extremely important in this context because it gives the aging power spectrum just mentioned in Section 1 [207]. In this context, there exist an attractive mini review article of Barkai [208]. Thus, we have this opportunity to mention that, beyond quantum mechanics, the blinking nano scale light emitters were discovered. To our surprise, this blinking is governed by the nonergodic statistics [209].

Lévy flights and walks proved to be very useful, still developing, theoretical tools in the field of cold atom optics. The former appeared as an anomalous diffusion related to the subrecoil laser cooling process [210–212]. The latter appeared in the context of experiment of Sagi et al., which gives the spreading of the cold atoms [213,214].

### 2.13.4 Search and foraging strategies

Searching and foraging is now located in the main stream of ecology, arousing the increasing interest of researchers. This is a young area of research inspired by pioneering papers on the Lévy flights versus Lévy walks by Shlesinger and Klafter [215] and on the Lévy flights of albatrosses [216] as well as on the optimality of the Lévy walk search [217]. This area contains extremely complex problems (e.g., animal search) which depend on many irreducible, unpredictable significant factors. Although the validity of the concept of the Lévy processes is still controversial in the ecological context, several comprehensive



**Fig. 1.** A report taken from Web of Science from the years 1978 (i.e., the year when the first review [19] of CTRW appeared) to 2015 (i.e., fifty years after publication of the Montroll and Weiss CTRW). Results were shown in the semi-logarithmic plot (sum of Times Cited without self-citations: 32333. H-index: 91). Notably, the exponential growth of citations is clearly seen.

monographs [218–220] support its spreading even beyond animals to humans and robotics.

Recently, Lévy walks became useful for communities researching the multiscale search and foraging strategies, and motility of living organisms from very primitive to extremely complex ones. It is a challenge to quantify their behaviour as it spans many scales (ranging from swimming bacteria to albatrosses which can soar even for hundreds of kilometers). Living organisms involve complicated interactions with environment containing their habitats. Nevertheless, Lévy walks are involved in the question of effectiveness of motility in the context of search and foraging strategies.

### 3 Financial applications of CTRW

Multidirectional applications of different versions of the CTRW formalism are a research trend which developed quite quickly in the past decade, resulting in an exponential growth of citations (see Fig. 1 for details). A prominent example of going far beyond the traditionally understood physics is an application of the CTRW formalisms by econophysicists; that is, in economics and finance. Their achievements have been presented in the review by Enrico Scalas in 2006 [221] and also in references [222–225]. Since then, a number of hopeful proposals appeared for the use of different variants of the CTRW to describe multiscale statistical properties of assets (e.g., their dependencies, memory, and correlations) quoted on financial markets [93,226] (and Refs. therein). Very recently, some universal properties or superscaling of superstatistics (i.e. scaling of scaling exponents) of returns for diverse financial markets were described by such a version of the renewal CTRW formalism, which contains thresholds separating explicit states of the walker activity from its hidden state [227]. It is not very surprising that this formalism well reproduced threshold empirical seismic activity data concerning various earthquakes.

## 4 General themes

Another general themes of great importance are the validity of the Green-Kubo and Einstein relations for the anomalous processes. There are a lot of works on these topics, for example, on the so called generalised Einstein relation vs. CTRW – see Barkai and Fleurov [228] and references therein. They discovered time scales (in the context of CTRW) at which deviations from the Einstein relation are expected to be large. Besides, they found some justification to the correctness of the Einstein relation for a model of symmetric random barriers. A more recent paper on the scale invariant Green-Kubo relation can be found [229] containing very instructive references.

## 5 Concluding remarks

The continuous-time random walk models define, in fact, two-state formalisms wherein the active state is defined by the walker displacements while the passive state is nothing else than waiting. These two essentially different states are present alternately during the journey of the walker. Obviously, one deals with extreme versions of the CTRW formalisms when the passive state of the walker vanishes at all or its displacements in the active state are instantaneous. The canonical CTRW formalism expanded for surprisingly many formulations and versions, where the canonical (original) version is only a very special case.

This year begins the next half-century of the continuous time random walk's mature life. The enclosed statistics present citations in each year together with a short report (taken from Web of Science). This confirms the incessant, great interest in the possibilities of CTRW methodology.

The present issue contains a collection of carefully selected papers on various deep and rich aspects of the CTRW and its promising inspirations prepared by scientists prominent in the field.

*Note added in proof.* Sadly we announce that on 14th February passed away George H. Weiss (1930–2017), an outstanding scientist and a great person.

We are deeply grateful to our colleagues Eli Barkai, Rudolf Hilfer, Peter Hänggi, Grégory Schehr, and Michael Shlesinger for very helpful comments and suggestions.

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