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Electric/magnetic dipole in an electromagnetic field: force, torque and energy

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Abstract. In this paper we collect the relativistic expressions for the force, torque and energy of a small electric/magnetic dipole in an electromagnetic field, which we recently obtained (A.L. Kholmetskii *et al.*, Eur. J. Phys. **33**, L7 (2011), Prog. Electromagn. Res. B **45**, 83 (2012), Can. J. Phys. **9**, 576 (2013)) and consider a number of subtle effects, characterized the behavior of the dipole in an external field, which seem interesting from the practical viewpoint.

1 Introduction

The problem of derivation of relativistic expressions for the force and torque experienced by a small electric/magnetic dipole in an external electromagnetic field was the subject of essential interest of researchers at the second half of the 20th century, and continues to attract attention up to date. The essential progress in this area was achieved with the derivation of the Bargmann-Michel-Telegdi (BMT) equation (1), which describes the time evolution of the particle's spin with the inclusion of Thomas precession [2]. At the same time, the approach used in the derivation of the BMT equation (the introduction of the spin four-vector with the vanishing time component in the proper frame of the particle [1,3]) is convenient to determine the motion of the particle's spin in its *proper* frame in terms of the external electromagnetic (EM) field measured in the laboratory. However, we remind that the transformation of the torque components is guided by the corresponding torque four-tensor [4], and one can check that its application does not allow to obtain an explicit analytical expression for the torque in the laboratory frame. At the same time, the derivation of the force and torque expressions, where all quantities are measured in the laboratory, has a principal importance in the consistent investigation of relativistic effects, related to the motion of an electric/magnetic dipole in an EM field.

In addition, before the discovery of the hidden momentum of magnetic dipoles [5], its contribution to the force and torque was not taken into account; moreover, it continues to be ignored in some modern publications, including textbooks (*e.g.*, [6]). One should mention that even in the past years, the problem of deriving the correct expression for the force and torque on a compact dipole faced a number of errors made by various authors (*e.g.*, [7–11]). Nevertheless, the correct Lorentz-invariant expression for the force on a moving electric/magnetic dipole, seemingly for the first time, has been achieved in our recent contribution [12], which opened the way for the derivation of the relativistic equations for the description of a torque on a moving dipole [13], and its electromagnetic energy [14]. Now it seems useful from the practical viewpoint to collect these equations altogether in a single paper and to discuss some subtle points, related to their physical meaning, which were not specially commented in the mentioned papers [12–14].

The obtained equations for the force $\boldsymbol{F}$, torque $\boldsymbol{T}$ and energy E of a small electrically neutral dipole with the proper electric $\boldsymbol{p}_0$ and magnetic $\boldsymbol{m}_0$ dipole moments, moving at velocity $\boldsymbol{\nu}$ in the external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$

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fields, are as follows [12–14]:

$$\boldsymbol{F}=\nabla(\boldsymbol{p}\cdot\boldsymbol{E})+\nabla(\boldsymbol{m}\cdot\boldsymbol{B})+\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p}\times\boldsymbol{B})-\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{m}\times\boldsymbol{E}), \tag{1}$$

$$\boldsymbol{T}=\boldsymbol{p}\times\boldsymbol{E}+\boldsymbol{m}\times\boldsymbol{B}+\frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{p}\times\boldsymbol{B})-\frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}), \tag{2}$$

$$E=-(\boldsymbol{p}\cdot\boldsymbol{E})-(\boldsymbol{m}\cdot\boldsymbol{B})-\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})\cdot\boldsymbol{\nu}+\frac{1}{c}(\boldsymbol{m}\times\boldsymbol{E})\cdot\boldsymbol{\nu}, \tag{3}$$

where

$$\boldsymbol{p}=\boldsymbol{p}_0-\frac{(\gamma-1)}{\gamma\boldsymbol{\nu}^2}(\boldsymbol{p}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu}+\frac{\boldsymbol{\nu}\times\boldsymbol{m}_0}{c}\,, \tag{4}$$

$$\boldsymbol{m}=\boldsymbol{m}_0-\frac{(\gamma-1)}{\gamma\boldsymbol{\nu}^2}(\boldsymbol{m}_0\cdot\boldsymbol{\nu})\boldsymbol{\nu}+\frac{\boldsymbol{p}_0\times\boldsymbol{\nu}}{c} \tag{5}$$

are, respectively, the electric and magnetic dipole moments of a moving dipole in the frame of observation [6,15,16], and $\gamma=(1-\boldsymbol{\nu}^2/c^2)^{-1/2}$ is the Lorentz factor. Here we stress that eq. (3) determines the electromagnetic energy of the dipole only, and does not include the fractions of energy absorbed (extracted) in the power supply and in a source of external magnetic field [14].

We point out that the quantities entering into eqs. (1)-(3) are defined in the frame of observation (laboratory frame), and these equations also can be expressed either via the quantities measured in the rest frame of a dipole, or via some combination of these quantities (*e.g.*, the proper electric and magnetic dipole moments, measured in the rest frame of a dipole, and electromagnetic fields measured in the laboratory. As we will see below, the latter combination is convenient for the analysis of the energy of the dipole).

In sect. 2 we discuss the physical meaning of eqs. (1)–(3), focusing mainly our attention on the expressions for torque (2) and energy (3) for a moving dipole, insofar as the expression for the force experienced by a dipole has been already discussed in the related publications [9,11,12]. Finally, we conclude in sect. 3.

2 Expressions for the force, torque and energy of a moving dipole: Origin and implications

2.1 Force on a moving electric/magnetic dipole in an electromagnetic field

In this subsection we address eq. (1) and remind that it originates from the Lorentz force law, applied to the electrically neutral compact bunch of charges [9] and also includes the force component due to time variation of the hidden momentum of the magnetic dipole (see, *e.g.* [3,5,17–19]). More specifically, the sum of the terms contaning $\boldsymbol{p}$ (*i.e.* the first and third terms) describes the Coulomb interaction of the charges of the electric dipole with the electric field, and the interaction of the convective currents of the dipole with the magnetic field. The sum of the terms contaning $\boldsymbol{m}$ (*i.e.* the second and fourth terms) is responsible for the interaction of the proper (closed) current of the magnetic dipole with the magnetic field, and also includes the hidden-momentum contribution.

A real situation might be more complicated, because the moving electric dipole develops the magnetic dipole moment, and the moving magnetic dipole possesses an electric dipole moment, see eqs. (4), (5). However, we skip the detailed analysis of the various terms in eq. (1), which can be found in the above-mentioned publications [9,11, 12], and would like to give answer to the question, which we ask time by time: why the Lorentz force law alone fails to describe the hidden-momentum contribution to the resultant force on a magnetic dipole, and the latter should be exogeneously added to the motional equation for a bunch of charges in an electromagnetic field?

The general answer to this question is straightforward: the Lorentz force law is relevant only for point-like charges, whereas an electric/magnetic dipole always represents some finite-size distribution of the charges. If so, an external electromagnetic field should induce some changes in its inner volume (polarization, magnetization, the appearance of mechanical stresses, etc.). The mentioned changes can be classified as some "secondary" effects emerging in the bulk of the dipole due to the external electromagnetic field. It was pointed out for the first time in ref. [5] that such secondary effects can be attributed to the appearance of the momentum component of the magnetic dipole, even if the dipole, as the whole, is at rest in the frame of observation; that is why this component was named as the "hidden" momentum. Moreover, it was further proven in ref. [17] that for a magnetic dipole, resting in an electric field, being created by a static-charge distribution, the hidden momentum of the dipole is exactly equal with the reverse sign to the interactional electromagnetic momentum of this configuration, and the time variation of the hidden momentum (which induces the appearance of force on the dipole) is exactly equal with the reverse sign to time variation of the field momentum. Therefore, the hidden momentum plays an important role in the energy-momentum balance for isolated systems, which include charges and magnetic dipoles.

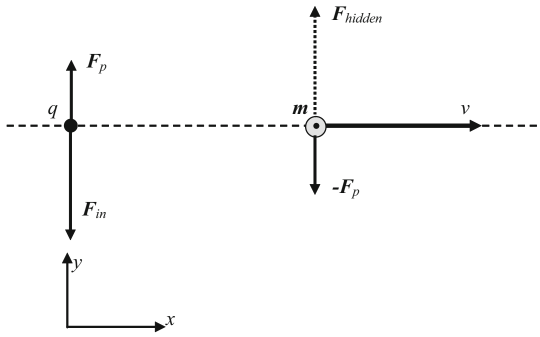

Fig. 1. Interaction of a resting charge q with a moving magnetic dipole $\boldsymbol{m}$. The dipole is moving without friction along the x-axis inside a thin insulating tube (not shown in the figure), and the charge is rigidly attached to the tube. The tube has a single degree of freedom to move along/opposite to the y-axis.

To demonstrate the validity of this statement, let us consider the following problem (see fig. 1). Let a small magnetic dipole with the proper magnetic dipole moment $\boldsymbol{m}\{0,0,m\}$ move inside a thin isolated tube oriented along the x-axis. Let a point-like charge q be rigidly fixed inside the same tube at the origin of the coordinates. We further asume that at the initial time moment $t=0$ the magnetic dipole has the x-coordinate X_1, and analyze the implementation of the energy-momentum conservation law, considering another time moment $t=T$, when the x-coordinate of the dipole becomes equal to X_2.

First consider the force, acting on the resting charge due to the moving magnetic dipole, which has two components:

– the interaction of the charge q with the induced electric field $\boldsymbol{E}_{in}=-\frac{\partial \boldsymbol{A}}{\partial t}$, where $\boldsymbol{A}$ is the vector potential of the dipole;
– the Coulomb interaction of the charge q with the electric dipole moment, developed by the moving magnetic dipole.

Assuming for simplicity a weak relativistic limit to the accuracy c^{-2}, which is sufficient for further analysis, we straightforwardly calculate both force components.

The vector potential of the magnetic dipole at the location of the charge has only the y-component

$$A_y=-\frac{m}{x^2}\,,$$

and the corresponding force on the charge due to the induced electric field

$$(F_{in})_y=-\frac{q}{c}\frac{\partial A_y}{\partial t}=-\frac{2qmv}{cx^3}\,. \tag{6}$$

Further, for the problem in question the electric dipole moment of the moving magnetic dipole lies in the negative y-direction, and is equal to

$$p_y=-\frac{vm}{c}\,. \tag{7}$$

Hence the force on the charge due to electric dipole (7) reads as

$$(F_p)_y=\frac{qmv}{cx^3}\,. \tag{8}$$

Considering now the reactive force on the magnetic dipole on behalf of the resting charge, and ignoring (for a moment) the force due to the time variation of the hidden momentum (*i.e.* the last term in eq. (1)), we get the single force component, describing the Coulomb force on the electric dipole (7) due to the resting charge. This force component lies in the negative y-direction and has the opposite sign with respect to the force (8). As a result, we have the exact balance of the Coulomb forces between charge and dipole, so that the non-compensated force (6) remains. Hence, while the dipole moves along the tube, the latter experiences the net force (6) and does accelerate in the negative y-direction during the motion of the magnetic dipole from the point X_1 to the point X_2.

This result already looks counter-intuitive. What is more, we can show that it violates the law of conservation of total energy (particles and fields). Indeed, the momentum transmitted to the tube during the time T is equal to

$$(P_{tube})_y=\int_0^T (F_{in})_y \mathrm{d}t=-\int_0^T \frac{2qmv}{cx^3}\mathrm{d}t=-\int_{X_1}^{X_2}\frac{2qm}{cx^3}\mathrm{d}x=-\frac{qm}{c}\left(\frac{1}{X_1^2}-\frac{1}{X_2^2}\right), \tag{9}$$

and the additional kinetic energy of the system "magnetic dipole, charge and tube" in the non-relativistic limit reads as

$$E_k = \frac{(P_{tube})_y^2}{2M}\,, \tag{10}$$

where M is the total rest mass of this configuration.

The change of kinetic energy (10) can be counteracted only by the corresponding change of the electromagnetic interactional energy between magnetic dipole and charge. This energy has only the electric component, caused by the interaction of the electric dipole moment (7) with the charge q. However, one can see that this energy is equal to zero at any time moment $0 < t < T$. Indeed, the scalar potential of the electric dipole moment (7) $\varphi = \boldsymbol{p}\cdot\boldsymbol{x}/x^3$ is equal to zero at the location of the charge q at any time moment. Thus, the energy (10) is taken from nothing.

This "paradox" is immediately resolved, when the hidden-momentum contribution (the last term of eq. (1)) is taken into account. Indeed, the electric field produced by the charge at the location of the magnetic dipole is equal to $E_x = q/x^2$, and

$$(F_{hidden})_y = -\frac{mv}{c}\frac{\partial E_x}{\partial x} = \frac{2qmv}{cx^3}\,. \tag{11}$$

Comparing now eqs. (6) and (11), we see that their sum is equal to zero and thus, no net force acts on the tube, so that it remains at rest during the entire time interval $0 < t < T$.

Thus, the total energy-momentum conservation law is perfectly fulfilled for the problem in fig. 1, when the force component due to the time variation of the hidden momentum is taken into account.

2.2 Torque on a moving electric/magnetic dipole in an electromagnetic field

The total torque on an electric/magnetic dipole can be presented in the form

$$\boldsymbol{T} = \int_V (\boldsymbol{r}\times\boldsymbol{f}_{total})\mathrm{d}V, \tag{12}$$

where $\boldsymbol{f}_{total}$ is the total force density, $\boldsymbol{r}$ is for the position vector of any point inside the dipole (where the center of mass of the dipole is located in the origin of the coordinates), and V stands for the volume of the dipole.

Like for the total force on a dipole, the force density $\boldsymbol{f}_{total}$ should be presented as the sum

$$\boldsymbol{f}_{total} = \boldsymbol{f}_L + \boldsymbol{f}_h, \tag{13}$$

where $\boldsymbol{f}_L$ is the density of the Lorentz force

$$\boldsymbol{f}_L = \rho_{total}\boldsymbol{E} + \frac{1}{c}\boldsymbol{j}_{total}\times\boldsymbol{B} \tag{14}$$

(see, *e.g.*, refs. [3,9]), and

$$\boldsymbol{f}_h = -\frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{M}\times\boldsymbol{E}) \tag{15}$$

is the density of the force due to the hidden-momentum contribution [13]. Here ρ_{total}, $\boldsymbol{j}_{total}$ are the total charge density and current density, respectively, and $\boldsymbol{M}$ is the magnetization.

We point out that the use of the partial time derivative in eq. (15) instead of the total time derivative for the corresponding force due to the time variation of the hidden momentum (the last term in eq. (1)) is related to the fact (usually omitted in the literature) that the measurement of magnetization (polarization) is carried out at a *fixed* point in the laboratory, whereas the measurement of the magnetic (electric) dipole moment of a moving bunch of charges is carried out for a volume V *co-moving* with this bunch.

Below we consider the case, when the free charges are absent, so that for a material medium we have the known relationships [3,9]:

$$\rho_{total} = -\nabla\cdot\boldsymbol{P}, \tag{16}$$

$$\boldsymbol{j}_{total} = \frac{\partial\boldsymbol{P}}{\partial t} + \nabla\times\boldsymbol{M}, \tag{17}$$

where $\boldsymbol{P}$ is the medium polarization.

Thus, combining eqs. (13)–(17), we derive the total force density in the form

$$\boldsymbol{f}_{total} = -(\nabla\cdot\boldsymbol{P})\boldsymbol{E} + (\nabla\times\boldsymbol{M})\times\boldsymbol{B} + \frac{1}{c}\frac{\partial\boldsymbol{P}}{\partial t}\times\boldsymbol{B} - \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{M}\times\boldsymbol{E}). \tag{18}$$

Substituting further eq. (18) into eq. (12), we get the expression for the total torque exerted on an electric/magnetic dipole in an electromagnetic field:

$$\boldsymbol{T}_{total} = \int_V \mathrm{d}V \left[\boldsymbol{r} \times \left(-(\nabla \cdot \boldsymbol{P})\boldsymbol{E} + (\nabla \times \boldsymbol{M}) \times \boldsymbol{B} + \frac{1}{c}\frac{\partial \boldsymbol{P}}{\partial t} \times \boldsymbol{B} - \frac{1}{c}\frac{\partial}{\partial t}(\boldsymbol{M} \times \boldsymbol{E})\right)\right]. \tag{19}$$

In order to clarify the physical meaning of the torque (19) and its components, we can simplify it in the case of a small dipole. This approximation implies, first of all, that the spatial variation of electric and magnetic fields at the location of the dipole can be practically ignored. This statement is expressed in the form of inequalities [11]:

$$\frac{\partial E_i}{\partial r_j}\Delta r_j \ll E_i, \qquad \frac{\partial B_i}{\partial r_j}\Delta r_j \ll B_i \quad (i,j = 1\ldots 3), \tag{20}$$

where Δr_j is the typical size of the dipole along the dimension j. Thus in the evaluation of the torque contributions of eq. (19) for *small* dipole, we can take the fields $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r})$, as functions of spatial coordinates, to be constant *within the volume* of such a dipole, because any terms, which include partial spatial derivatives of the electric and magnetic field become negligible due to the inequalities (20). We notice that the approximation of constant fields cannot be adopted in the calculation of the force (1) acting on a small dipole, because some of the force components are vanishing at $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{B}(\boldsymbol{r}) = \text{const.}$, and we have to involve the terms, containing their spatial derivatives (see, *e.g.*, [9,11]). At the same time, when the motion of dipoles is considered, the variation of electric and magnetic fields along *a trajectory* of a moving dipole, in general, cannot be ignored, so that the approximation of a constant (as a function of spatial coordinates) field is relevant for a small volume of dipole only, and cannot be applied to the entire space.

So, we consider a small electric/magnetic dipole, moving at velocity $\boldsymbol{\nu}$ in the external electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields of a laboratory frame. For simplicity we assume that the electric $\boldsymbol{p} = \int_V \boldsymbol{P}\mathrm{d}V$ and magnetic $\boldsymbol{m} = \int_V \boldsymbol{M}\mathrm{d}V$ dipole moments are constant values in the rest frame of the dipole. With these limitations we obtain for the first term in the integrand of eq. (19)

$$-\int_V \boldsymbol{r} \times (\nabla \cdot \boldsymbol{P})\boldsymbol{E}\mathrm{d}V = \boldsymbol{p} \times \boldsymbol{E}, \tag{21}$$

which describes the contribution to the torque due to the Coulomb interation. The derivation of this equation is presented in ref. [13].

Next we evaluate the torque component due to interaction of the proper current of a dipole with a magnetic field (second term in the integrand of eq. (19)). Using the vector identities $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ and $\boldsymbol{a} \times (\nabla \times \boldsymbol{b}) = \nabla(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{b} \times (\nabla \times \boldsymbol{a}) - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} - (\boldsymbol{b} \cdot \nabla)\boldsymbol{a}$, this term can be presented in the form

$$\begin{aligned}\int_V \boldsymbol{r} \times ((\nabla \times \boldsymbol{M}) \times \boldsymbol{B})\mathrm{d}V &= -\int_V \boldsymbol{r} \times (\boldsymbol{B} \times (\nabla \times \boldsymbol{M}))\mathrm{d}V \\ &= -\int_V \boldsymbol{r} \times (\nabla(\boldsymbol{B} \cdot \boldsymbol{M}) - (\boldsymbol{B} \cdot \nabla)\boldsymbol{M} - (\boldsymbol{M} \cdot \nabla)\boldsymbol{B} - \boldsymbol{M} \times (\nabla \times \boldsymbol{B}))\mathrm{d}V \\ &= \int_V [\boldsymbol{r} \times (\boldsymbol{B} \cdot \nabla)\boldsymbol{M}]\mathrm{d}V. \end{aligned} \tag{22}$$

Here we have taken into account that the integral $\int_V [\boldsymbol{r} \times \nabla(\boldsymbol{B} \cdot \boldsymbol{M})]\mathrm{d}V$ can be transformed into the surface integral, where the magnetization $\boldsymbol{M}$ is vanishing; also we notice that in the adopted approximation ($\boldsymbol{B} \approx \text{const.}$ inside the volume of the dipole), the terms $(\boldsymbol{M} \cdot \nabla)\boldsymbol{B}$ and $\boldsymbol{M} \times (\nabla \times \boldsymbol{B})$ are vanishing too. Integrating by parts the remaining integral in eq. (22), we obtain [13]

$$\int_V [\boldsymbol{r} \times (\boldsymbol{B} \cdot \nabla)\boldsymbol{M}]\mathrm{d}V = \boldsymbol{\mu} \times \boldsymbol{B}. \tag{23}$$

Further, we evaluate the term responsible for the interaction of the convective currents of a dipole with a magnetic field (third term in the integrand of eq. (19)). Here we notice that due to the adopted constancy of the proper electric dipole moment, we get $\partial \boldsymbol{P}/\partial t = 0$ in the rest frame of the dipole. However, for a moving dipole the stationary distribution of its charges yields $\mathrm{d}\boldsymbol{P}/\mathrm{d}t = 0$, and hence $\partial \boldsymbol{P}/\partial t = -(\boldsymbol{\nu} \cdot \nabla)\boldsymbol{P}$. With the latter equality we derive

$$-\frac{1}{c}\int_V \boldsymbol{r} \times ((\boldsymbol{\nu} \cdot \nabla)\boldsymbol{P} \times \boldsymbol{B})\mathrm{d}V = -\frac{1}{c}\int_V \boldsymbol{r} \times ((\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{P} \times \boldsymbol{B}))\mathrm{d}V = \frac{1}{c}\boldsymbol{\nu} \times (\boldsymbol{p} \times \boldsymbol{B}). \tag{24}$$

The proof of eq. (24) can be found in ref. [13].

Addressing now the last term of the integrand of eq. (19) (the hidden-momentum contribution), and taking into account that for the stationary magnetization $\frac{\partial}{\partial t}(\boldsymbol{M} \times \boldsymbol{E}) = -(\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{M} \times \boldsymbol{E})$, we obtain by analogy with eq. (24)

$$\boldsymbol{T}_h = -\frac{1}{c}\int_V \left(\boldsymbol{r} \times \frac{\partial}{\partial t}(\boldsymbol{M} \times \boldsymbol{E})\right)\mathrm{d}V = \frac{1}{c}\int_V (\boldsymbol{r} \times (\boldsymbol{\nu} \cdot \nabla)(\boldsymbol{M} \times \boldsymbol{E}))\mathrm{d}V = -\frac{1}{c}\boldsymbol{\nu} \times (\boldsymbol{m} \times \boldsymbol{E}). \tag{25}$$

Finally, substituting eqs. (21), (23)-(25) into eq. (19), we derive the expression for the total torque exerted on a small dipole in an electromagnetic field [13]:

$$\boldsymbol{T}_{total} = \boldsymbol{p}\times\boldsymbol{E} + \boldsymbol{m}\times\boldsymbol{B} + \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{p}\times\boldsymbol{B}) - \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}), \tag{26}$$

where all quantities are evaluated in the laboratory frame.

The first and second terms on the rhs of eq. (26) look similar to the corresponding known terms for the torque on a resting dipole (see, *e.g.* [15]), though now they include the electric $\boldsymbol{p}$ and magnetic $\boldsymbol{\mu}$ dipole moments of a moving dipole. The third term is responsible for the interaction of the convective currents of a dipole with the magnetic field, while the fourth term stands for the hidden-momentum contribution to the torque on a dipole. To our recollection, the last two terms in eq. (26) seem not to have been reported before publication of our paper [13].

Now it is worth comparing our equation (26) with the results obtained earlier by Namias. In ref. [7] he considered separately a torque on an electric dipole ($\boldsymbol{p}_0 \neq 0$, $\boldsymbol{m}_0 = 0$) and a torque on a magnetic dipole ($\boldsymbol{p}_0 = 0$, $\boldsymbol{m}_0 \neq 0$). Applying the electric-charge model and magnetic-charge model, he subsequently derived the expressions for the torque as follows (in Gaussian units):

$$\boldsymbol{T}_p = \boldsymbol{p}\times\boldsymbol{E} + \frac{1}{c}\boldsymbol{p}\times(\boldsymbol{\nu}\times\boldsymbol{B})\,(\boldsymbol{p}_0\neq 0, \boldsymbol{m}_0 = 0), \tag{27a}$$

and

$$\boldsymbol{T}_m = \boldsymbol{m}\times\boldsymbol{B} - \frac{1}{c}\boldsymbol{m}\times(\boldsymbol{\nu}\times\boldsymbol{E})\,(\boldsymbol{p}_0 = 0, \boldsymbol{m}_0\neq 0). \tag{27b}$$

We stress that the summation of eqs. (27a) and (27b) (to try to get the general expression for the torque at $\boldsymbol{p}_0 \neq 0$, $\boldsymbol{\mu}_0 \neq 0$) is in general incorrect, because in this case we fall outside the scope of validity of each of these equations. Thus, in order to make a comparison of eq. (26) with the results of ref. [7], we can apply eq. (26) to the particular cases $\boldsymbol{p}_0 \neq 0$, $\boldsymbol{m}_0 = 0$ and $\boldsymbol{p}_0 = 0$, $\boldsymbol{m}_0 \neq 0$. In the first case eq. (26) yields

$$\boldsymbol{T}_{total} = \boldsymbol{p}\times\boldsymbol{E} + \frac{1}{c}(\boldsymbol{p}_0\times\boldsymbol{\nu})\times\boldsymbol{B} + \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{p}\times\boldsymbol{B}) - \frac{1}{c^2}\boldsymbol{\nu}\times((\boldsymbol{p}_0\times\boldsymbol{\nu})\times\boldsymbol{E}) \quad (\boldsymbol{p}_0\neq 0, \boldsymbol{m}_0 = 0), \tag{28}$$

where we have taken into account that the moving electric dipole develops the magnetic dipole moment (the last term of eq. (5)).

We see that, in general, our equation (28) differs from the Namias equation (27a) in terms of the order $(\nu/c)^2$. In particular, the last term of eq. (28) indicates that the hidden momentum should be also associated with the magnetic dipole moment, arising due to the motion of the electric dipole. The physical meaning of this term, which is missing both in the electric-charge model and in the magnetic-charge model, is discussed in more detail in ref. [13] (see also footnote[1] below).

Next we apply eq. (26) to the analysis of some physical problems, which could be interesting to the readers. First consider a resolution of the paradox by Mansuripur via a direct application of this equation.

In a recent letter [10] Mansuripur questioned the correctness of the Lorentz force law in material media, arguing that the force law by Einstein and Laub [20,21] must be adopted instead, insofar as only the latter law provides the implementation of the momentum conservation law and exhibits compatibility with special relativity. As a central point of his argumentation, the author of [10] considers the interaction between a point-like charge q and a point-like magnetic dipole $\boldsymbol{m}$, which are both at rest with respect to each other in a reference frame K', and the direction of $\boldsymbol{m}$ is orthogonal to the line joining charge and dipole (the z-axis). In this frame the mutual force between charge and magnetic dipole is equal to zero, and no torque is exerted on the dipole by the charge at rest. Then he considers the situation, where the frame K' is moving at constant velocity V along the z-axis of the laboratory frame K, and shows that in this frame the forces acting on charge and dipole remain zero. However, one can check that the Lorentz force law (14) yields a non-vanishing torque exerted on the moving magnetic dipole by the moving charge. This obviously represents a non-adequate result, since in the proper frame of charge and dipole K', the torque is equal to zero. On the other hand, when the Einstein-Laub formula is applied, both the force and the torque are equal to zero in the frames K' and K. Based on this result, Mansuripur concluded that for material media the Lorentz force law must be abandoned in favor of the Einstein-Laub law.

The subsequent comments on this paper [22–25] invalidated the conclusion by Mansuripur and showed that the approach based on the Lorentz force law with taking into account the hidden momentum of the magnetic dipole yields the vanishing torque in both K' and K frames, which resolves the paradox. Here we mention that Mansuripur was well aware of the hidden-momentum contribution to the force on a magnetic dipole (the last term of eq. (1)), but decided, perhaps, that this contribution could be omitted in the problem he considered, because in his configuration the force component due to the time variation of the hidden momentum is exactly equal to zero. However, he missed the fact that the torque on the magnetic dipole due to the hidden-momentum contribution is not zero. In general, the aspect of non-vanishing torque on some body with zero total force on this body is trivial in classical mechanics. However,

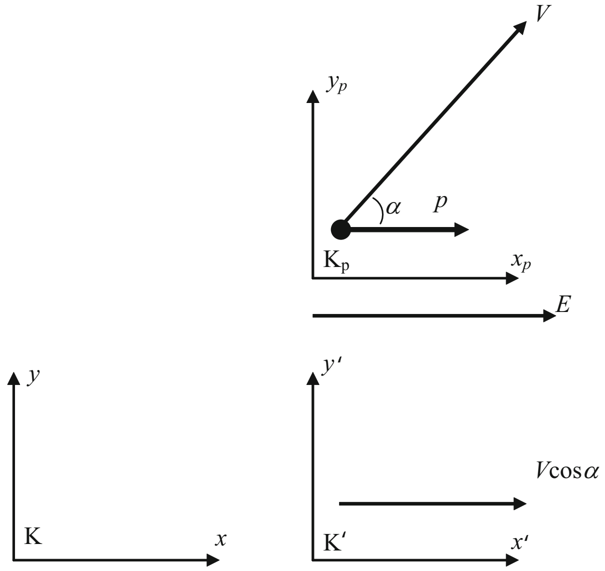

Fig. 2. Small electric dipole with its proper electric dipole moment $\boldsymbol{p}_0$, oriented along the x_p-axis of its rest frame K_p, is moving in the plane xy of the laboratory frame K at constant velocity $\boldsymbol{V}$, constituting an angle α with the x-axis. We want to calculate the torque exerted on the dipole due to the constant electric field $\boldsymbol{E}$, lying in the positive x-direction, in the frame K and in another inertial frame K', moving along the x-axis with constant velocity $V\cos\alpha$.

this fact was overlooked by the author of [10] and this fact was not stressed in the comments [22–25] too, with respect to the force and torque due to the hidden momentum contribution.

Indeed, applying eq. (26) to the mentioned problem of ref. [10] about the interaction of a point-like charge and a magnetic dipole, we obtain in the laboratory frame K

$$\boldsymbol{T}_{total} = \boldsymbol{p}\times\boldsymbol{E} - \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}), \tag{29}$$

where the electric dipole moment $\boldsymbol{p} = \frac{1}{c}(\boldsymbol{\nu}\times\boldsymbol{m})$ appears due to the relativistic polarization of the moving magnetic dipole. Further on, the term $\boldsymbol{p}\times\boldsymbol{E}$ can be presented in the form

$$\boldsymbol{p}\times\boldsymbol{E} = \frac{1}{c}(\boldsymbol{\nu}\times\boldsymbol{m})\times\boldsymbol{E} = -\frac{1}{c}\boldsymbol{E}\times(\boldsymbol{\nu}\times\boldsymbol{m}) = \frac{1}{c}\boldsymbol{m}\times(\boldsymbol{E}\times\boldsymbol{\nu}) + \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{E}\times\boldsymbol{m}) = \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{E}\times\boldsymbol{m}), \tag{30}$$

where we have used the Jacobi identity and taken into account that for the problem in question the vectors $\boldsymbol{\nu}$, $\boldsymbol{E}$ are parallel to each other at the location of the point-like dipole, and $\boldsymbol{E}\times\boldsymbol{\nu} = 0$. Hence, substituting eq. (30) into eq. (29), we get $\boldsymbol{T}_{total} = 0$ in the laboratory frame K, as it should be according to the relativistic requirements.

Thus, in contrast to the claim made in ref. [10], we conclude that the approach based on the Lorentz force law complemented by the hidden-momentum contribution, yielding eq. (26), is fully compatible with the relativistic requirements and the momentum conservation law.

Here we again highlight the Lorentz invariance of the novel expression (26), which thus incorporates all the relativistic effects of special relativity. For example, let us show that in successive space-time transformations, eq. (26) along with the relativistic transformations for the electric and magnetic dipole moments (4), (5) yields relativistically adequate results with the inclusion of the Thomas-Wigner rotation [26].

Consider the following problem (see fig. 2). The electric dipole with the proper electric dipole moment $\boldsymbol{p}_0$, oriented along the x_p-axis of its rest frame K_p, is moving at constant velocity $\boldsymbol{V}$, constituting an angle α with the x-axis of the laboratory frame K. The static electric field $\boldsymbol{E}$, as measured in the laboratory frame, lies along the x-axis. We want to calculate the torque exerted on the dipole in the frame K, and also in another inertial reference frame K', which moves along the x-axis of K with constant velocity $V\cos\alpha$.

First of all, we observe that the motion of the frame K' along the lines of the constant electric field $\boldsymbol{E}$ signifies that this field is the same in both K and K' frames, and no magnetic field emerges in these frames. Hence, according to eq. (26), the expression for the torque on the dipole has the same form in K and K' frames:

$$\boldsymbol{T}_{total} = \boldsymbol{p}\times\boldsymbol{E} - \frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}), \tag{31}$$

where, however, the parameters $\boldsymbol{\nu}$, $\boldsymbol{p}$ and $\boldsymbol{m}$ are different for observers in these frames.

In particular, for a laboratory observer we have $\boldsymbol{\nu} = \boldsymbol{V}\{V\cos\alpha, V\sin\alpha, 0\}$. Thus, substituting this velocity into eqs. (4), (5), and taking into account that $\boldsymbol{p}_0\{p_0, 0, 0\}$, $\boldsymbol{m}_0 = 0$, we obtain

$$p_x = p_0 - \frac{(\gamma-1)}{\gamma V^2}(p_0 V_x)V_x = p_0\left[1 - \frac{(\gamma-1)}{\gamma}\cos^2\alpha\right], \tag{32a}$$

$$p_y = -\frac{(\gamma-1)}{\gamma V^2}(p_0 V_x)V_y = -p_0\frac{(\gamma-1)}{\gamma}\sin\alpha\cos\alpha, \tag{32b}$$

$$m_z = \frac{p_0 V_y}{c} = \frac{p_0 V\sin\alpha}{c}, \tag{32c}$$

where $\gamma = (1 - V^2/c^2)^{-1/2}$.

Equation (32b) shows the appearance of the non-vanising y-component of the electric dipole moment on the y-axis of the frame K, so that the vector $\boldsymbol{p}$ is no longer parallel to the x-axis of this frame. From the physical viewpoint, this result is explained by the relativistic contraction of the moving scale along the direction of motion. Indeed, the component of the length of the dipole on vector $\boldsymbol{V}$ is contracted $1/\gamma$ times, while the component of the length orthogonal to $\boldsymbol{V}$ remains unchanged. Hence the dipole as a whole is turned with respect to the x-axis at a negative angle φ, defined via the equation

$$\tan\varphi = -\frac{p_y}{p_x} = -\frac{\frac{(\gamma-1)}{\gamma}\sin\alpha\cos\alpha}{1 - \frac{(\gamma-1)}{\gamma}\cos^2\alpha}.$$

To the accuracy of calculations c^{-2}, which is sufficient for further analysis, this equation yields

$$\varphi \approx -\frac{V^2}{2c^2}\sin\alpha\cos\alpha = -\frac{V_x V_y}{2c^2}. \tag{33}$$

Here the negative sign of the angle φ corresponds to the clockwise rotation.

Correspondingly, the vector $\boldsymbol{p}$ is no longer parallel to the electric field $\boldsymbol{E}$ for a laboratory observer, and the Coulomb force induces the non-vanishing torque component

$$(\boldsymbol{T}_{Coulomb})_z = (\boldsymbol{p}\times\boldsymbol{E})_z = -p_y E \approx p_0 E\frac{V_x V_y}{2c^2} \tag{34}$$

in the positive z-direction.

At the same time, the Coulomb contribution (34) does not yet determine the total torque on the dipole. The moving electric dipole develops the magnetic dipole moment, which in our case is determined by eq. (32c). According to eq. (31), it gives one more component of the torque:

$$\boldsymbol{T}_{hidden} = -\frac{1}{c}\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}). \tag{35}$$

Combining eqs. (35) and (32c), we derive

$$(\boldsymbol{T}_{hidden})_z = -\frac{1}{c}(\boldsymbol{\nu}\times(\boldsymbol{m}\times\boldsymbol{E}))_z = -\frac{V_x}{c}m_z E = -p_0 E\frac{V_x V_y}{c^2}, \tag{36}$$

and this torque component lies in the negative z-direction.

Thus, the total torque on the moving electric dipole is equal to the sum of eqs. (34), (36), and lies in the negative z-direction:

$$(\boldsymbol{T}_{total})_z \approx -p_0 E\frac{V_x V_y}{2c^2}. \tag{37}$$

Here it is interesting to note that a laboratory observer sees the moving electric dipole to be turned in the clockwise direction with respect to the lines of electric field $\boldsymbol{E}$, and intuitively he may expect the torque on the dipole in the pozitive z-direction. However, the moving electric dipole has the magnetic dipole moment and, like an "actual" magnetic dipole, possesses the hidden momentum, interacting with the electric field and inducing the torque component (36) in the negative z-direction. As a result, the total torque on the dipole appears to be negative too.

This result looks somewhat paradoxical. However, its adequacy from the relativistic viewpoint can be demonstrated via calculation of the torque on the dipole for an observer in the frame K'.

In this frame the velocity of dipole $\boldsymbol{V}'$ is parallel to the y'-axis, and the electric field $\boldsymbol{E}' = \boldsymbol{E}$ is parallel to the x'-axis. In these conditions the second term on the rhs of eq. (31) (the hidden-momentum contribution) is vanishing:

$$\boldsymbol{T}'_{hidden} = -\frac{1}{c}\boldsymbol{V}' \times (\boldsymbol{m}' \times \boldsymbol{E}') = \frac{1}{c}\boldsymbol{m}'(\boldsymbol{V}' \cdot \boldsymbol{E}') - \frac{1}{c}\boldsymbol{E}'(\boldsymbol{m}' \cdot \boldsymbol{V}') = 0,$$

where we have taken into account the orthogonality of vectors $\boldsymbol{V}'$, $\boldsymbol{E}'$ and $\boldsymbol{m}'$ (which lie along the z'-axis). Hence, only the Coulomb contribution to the torque (the first term on the rhs of eq. (31)) is fixed in the frame K'. In order to determine its value, we have to find the components of the electric dipole moment $\boldsymbol{p}'$ for an observer in K'. Using eqs. (32a-c), which give the components of the electric and magnetic dipole moments in K, and taking into account that the frame K is moving in K' at the velocity $-V\cos\alpha$ along the x'-axis, we derive via eq. (4)

$$p'_x = p_x - \frac{(\gamma'-1)}{\gamma'(V\cos\alpha)^2}(p_x V\cos\alpha)V\cos\alpha = p_x\left[1 - \frac{(\gamma'-1)}{\gamma'}\right]$$

$$\approx p_x\left(1 - \frac{V^2\cos^2\alpha}{2c^2}\right) \approx p_0\left(1 - \frac{V^2\cos^2\alpha}{c^2}\right), \tag{38a}$$

$$p'_y = p_y + \frac{1}{c}(m_z V\cos\alpha) = -p_0\frac{(\gamma-1)}{\gamma}\sin\alpha\cos\alpha + p_0\frac{V^2\sin\alpha\cos\alpha}{c^2}$$

$$\approx -p_0\frac{V^2\sin\alpha\cos\alpha}{2c^2} + p_0\frac{V^2\sin\alpha\cos\alpha}{c^2} = p_0\frac{V^2\sin\alpha\cos\alpha}{2c^2} = p_0\frac{V_xV_y}{2c^2}\,, \tag{38b}$$

where we designated $\gamma' = (1 - \frac{V^2\cos^2\alpha}{c^2})^{-1/2} \approx 1 + \frac{V^2\cos^2\alpha}{2c^2}$.

Thus, the angle φ' between the vector $\boldsymbol{p}'$ and x'-axis of the frame K' is equal to

$$\varphi' \approx \frac{p'_y}{p'_x} = \frac{V_xV_y}{2c^2}\,, \tag{39}$$

and happens to be positive (the rotation is in the counterclockwise direction). Correspondingly, the Coulomb component of the torque on the dipole (which represents the total torque in the frame K'), is equal to

$$(\boldsymbol{T}'_{total})_z = (\boldsymbol{p}' \times \boldsymbol{E}')_z = -p_y E \approx -p_0 E\frac{V_xV_y}{2c^2} \tag{40}$$

and lies in the negative z'-direction.

Thus, we find that the torque on the moving electric dipole, as seen in the frames K and K', is directed opposite to the z-axis; it is the same in the adopted accuracy of calculations c^{-2} (compare eqs. (37) and (40)), and induces a clockwise rotation of the dipole for the observers in these frames.

We see a full adequacy of this result from the relativistic viewpoint[1]. It is also important to notice that the rotation of the vector $\boldsymbol{p}'$ at the angle φ' with respect to the x'-axis concurrently means the same rotation of the x_p-axis of K_p with respect to the x'-axis of K'. Furthermore, considering the electric dipole $\boldsymbol{p}_0$, originally oriented along the y_p-axis of its rest frame K_p, and acting in a similar way, we can show that the y_p-axis of K_p is seen in the frame K' to be turned at the same angle φ' with respect to the y'-axis. Thus we conclude that for the motion diagram of fig. 2, the axes of the frames K_p and K' are no longer parallel to each other and experience a common rotation at the angle φ'. This is the known relativistic effect of the Thomas-Wigner rotation, appearing in successive space-time transformations with non-collinear velocities.

2.3 Energy of electric/magnetic dipole in an electromagnetic field

Deriving an expression for the energy of electrically neutral electric/magnetic dipole, we assume for simplicity the constancy of its proper electric $\boldsymbol{p}_0$ and magnetic $\boldsymbol{m}_0$ dipole moments. Furthermore, at the initial time moment $t = 0$ let the dipole move with constant velocity $\boldsymbol{\nu}$, as measured in the laboratory frame, being far from the region with the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields. When the dipole is approaching the field region, it begins to experience the

[1] Here we mention that the application of Namias equations (27a), (27b) to the calculation of a torque on a moving electric dipole in the frame K gives a *positive* value of the z-component of this torque (inducing a counterclockwise rotation of the dipole), which thus is relativistically non-adequate.

electromagnetic force $\boldsymbol{F}$ according to (1). In these conditions, we assume the presence of some external force $\boldsymbol{F}_{ext} = -\boldsymbol{F}$, which exactly counteracts the force (1), so that the velocity of the dipole remains constant in the entire region with electromagnetic field, and its non-electromagnetic energy $\gamma M_0 c^2$ (where M_0 is the rest mass of the dipole) does not change with time. Hence, the work done to the electric/magnetic dipole in the external electric and magnetic fields is equal to the electromagnetic energy of the dipole in this field. At any time moment t, this work is equal to

$$E = \int_0^t \boldsymbol{F}_{ext} \cdot \boldsymbol{\nu} \mathrm{d}t = -\int_0^t \boldsymbol{F} \cdot \boldsymbol{\nu} \mathrm{d}t. \tag{41}$$

Substituting eq. (1) into eq. (41), we get the electromagnetic energy of the dipole:

$$E_{EM} = -\int_0^t \boldsymbol{F} \cdot \boldsymbol{\nu} \mathrm{d}t = -\int \nabla(\boldsymbol{p} \cdot \boldsymbol{E}) \cdot \mathrm{d}\boldsymbol{s} - \int \nabla(\boldsymbol{m} \cdot \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{s} - \frac{1}{c}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} \times \boldsymbol{B}) \cdot \boldsymbol{\nu} \mathrm{d}t + \frac{1}{c}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{m} \times \boldsymbol{E}) \cdot \boldsymbol{\nu} \mathrm{d}t,$$

or

$$E_{EM} = -(\boldsymbol{p} \cdot \boldsymbol{E}) - (\boldsymbol{m} \cdot \boldsymbol{B}) - \frac{1}{c}(\boldsymbol{p} \times \boldsymbol{B}) \cdot \boldsymbol{\nu} + \frac{1}{c}(\boldsymbol{m} \times \boldsymbol{E}) \cdot \boldsymbol{\nu}, \tag{42}$$

where we designated $\mathrm{d}\boldsymbol{s} = \boldsymbol{\nu}\mathrm{d}t$ the element of the path of the dipole.

As we have mentioned above, due to the constancy of $\boldsymbol{\nu}$, eq. (42) determines the electromagnetic energy of the electric/magnetic dipole in an external electromagnetic field, though we point out that this energy still does not include the fraction of energy absorbed (or extracted) in a power supply, maintaining the constancy of $\boldsymbol{m}_0$ for the dipole in question. One should add that this equation also does not include the energy absorbed (or extracted) in a source of external magnetic field, if we assume that the motion of the dipole does not cause any variation of the external field.

We see that at $v = 0$, eq. (42) is transformed into the known expression for the electromagnetic energy of a resting electric/magnetic dipole, *i.e.*

$$(E_{EM})_0 = -(\boldsymbol{p}_0 \cdot \boldsymbol{E}) - (\boldsymbol{m}_0 \cdot \boldsymbol{B}). \tag{43}$$

For a moving dipole, in addition to the replacements $\boldsymbol{p}_0 \to \boldsymbol{p}$, $\boldsymbol{m}_0 \to \boldsymbol{m}$ in the firsts two terms of eq. (42) in comparison with eq. (43), two new terms with explicit dependence on $\boldsymbol{\nu}$, *i.e.* $-\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})\cdot\boldsymbol{\nu}$ and $\frac{1}{c}(\boldsymbol{m}\times\boldsymbol{E})\cdot\boldsymbol{\nu}$, do emerge. The energy term $\frac{1}{c}(\boldsymbol{m}\times\boldsymbol{E})\cdot\boldsymbol{\nu}$, being related to the hidden-momentum contribution to the energy of a moving dipole, for the first time has been introduced by Hnizdo [27] based on the requirement of Lorentz invariance of classical electrodynamics with the inclusion of hidden momentum. To our recollection, the energy term $-\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})\cdot\boldsymbol{\nu}$ was not reported earlier and it is directly related to the time variation of the cross product $-\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})$, which we suggested to name as "latent" momentum of electric dipole [14].

For further progress in the analysis of the energy of the electric/magnetic dipole in an electromagnetic field, we write eq. (42) via the proper electric and magnetic dipole moments, using eqs. (4) and (5). Substituting these equations into equation (42), we derive after straightforward manipulations [14]

$$E_{EM} = -\frac{(\boldsymbol{p}_0 \cdot \boldsymbol{E})}{\gamma^2} - \frac{(\boldsymbol{m}_0 \cdot \boldsymbol{B})}{\gamma^2} - \frac{(\boldsymbol{p}_0 \cdot \boldsymbol{\nu})(\gamma-1)}{\gamma^2 v^2}(\boldsymbol{\nu} \cdot \boldsymbol{E}) - \frac{(\boldsymbol{m}_0 \cdot \boldsymbol{\nu})(\gamma-1)}{\gamma^2 v^2}(\boldsymbol{\nu} \cdot \boldsymbol{B}). \tag{44}$$

Finally, the energy (44) can be also expressed through the electric $\boldsymbol{E}_0$ and magnetic $\boldsymbol{B}_0$ fields in the rest frame of the dipole, when we use the relativistic transformations of the fields (see, *e.g.* [3]):

$$\begin{aligned} \boldsymbol{E} &= \gamma \boldsymbol{E}_0 - \frac{(\gamma-1)}{v^2}(\boldsymbol{E}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} - \gamma\frac{\boldsymbol{\nu} \times \boldsymbol{B}_0}{c}, \\ \boldsymbol{B} &= \gamma \boldsymbol{B}_0 - \frac{(\gamma-1)}{v^2}(\boldsymbol{B}_0 \cdot \boldsymbol{\nu})\boldsymbol{\nu} + \gamma\frac{\boldsymbol{\nu} \times \boldsymbol{E}_0}{c}. \end{aligned} \tag{45}$$

Substituting eqs. (45) into eq. (44), we obtain (for the details of calculations see [14])

$$E_{EM} = -\frac{1}{\gamma}\left(\boldsymbol{p}_0 \cdot \boldsymbol{E}_0 + \boldsymbol{m}_0 \cdot \boldsymbol{B}_0 + \frac{(\boldsymbol{p}_0 \times \boldsymbol{B}_0) \cdot \boldsymbol{\nu}}{c} - \frac{(\boldsymbol{m}_0 \times \boldsymbol{E}_0) \cdot \boldsymbol{\nu}}{c}\right). \tag{46}$$

The obtained eq. (46) can be considered as the relativistic transfomation of the electromagnetic energy of the electric/magnetic dipole, which can be rewritten in the symbolic form as follows:

$$E_{EM} = \frac{1}{\gamma}((E_{EM})_0 + (\boldsymbol{p}_{l0} + \boldsymbol{p}_{h0}) \cdot \boldsymbol{\nu}), \tag{47}$$

where the proper electromagnetic energy of the dipole $(E_{EM})_0$ is defined by eq. (43), $\boldsymbol{p}_{l0} = -\boldsymbol{p}_0 \times \boldsymbol{B}_0/c$ is the proper latent momentum, and $\boldsymbol{p}_{h0} = \boldsymbol{m}_0 \times \boldsymbol{E}_0/c$ is the proper hidden momentum of the dipole. Here it is worth stressing that eq. (47) does not coincide with the relativistic transformation of energy-momentum, because E_{EM} does not determine the *total* energy of the electric/magnetic dipole, as we mentioned above.

For practical applications, eq. (44) is the most convenient. Indeed, this equation operates with the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields directly measured in a laboratory, as well as with the proper electric $\boldsymbol{p}_0$ and magnetic $\boldsymbol{m}_0$ dipole moments, usually considered as known quantities for a given bunch of charges.

Analyzing the structure of eq. (44), we reveal its important property: the energy of the electric dipole depends solely on the presence of an electric field $\boldsymbol{E}$, and does not depend on the magnetic field $\boldsymbol{B}$. Analogously, the energy of the magnetic dipole depends solely on the presence of a magnetic field $\boldsymbol{B}$, and does not depend on the electric field $\boldsymbol{E}$. In particular, we observe that for an electric dipole, moving in a time-independent magnetic field with vanishing electric field ($\boldsymbol{E} = 0$), the energy of the dipole is equal to zero at any time moment. This result reflects the known fact that the energy of any bunch of charges, moving in a magnetic field, remains unchanged, since the magnetic Lorentz force component is always orthogonal to the velocity of the charges.

It is interesting to notice that the symmetric situation occurs with respect to a magnetic dipole, moving in a time-independent electric field with vanishing magnetic field ($\boldsymbol{B} = 0$): the energy of such a magnetic dipole is equal to zero too.

From the physical viewpoint, the constancy of energy of a magnetic dipole in a time-independent electric field can be explained by the fact that its hidden energy (the last term in eq. (3))

$$E_h = \frac{(\boldsymbol{m} \times \boldsymbol{E}) \cdot \boldsymbol{\nu}}{c} \tag{48}$$

is exactly balanced by the electric energy

$$U_{rel} = -\frac{(\boldsymbol{\nu} \times \boldsymbol{m}_0) \cdot \boldsymbol{E}}{c}\,, \tag{49}$$

associated with the relativistic polarization of a moving magnetic dipole, which leads to the appearance of the electric dipole moment $\boldsymbol{p} = \frac{\boldsymbol{\nu} \times \boldsymbol{m}_0}{c}$. Indeed, summing up eqs. (48) and (49), we have

$$\begin{aligned} E_h + U_{rel} &= \frac{(\boldsymbol{m} \times \boldsymbol{E}) \cdot \boldsymbol{\nu}}{c} - \frac{(\boldsymbol{\nu} \times \boldsymbol{m}_0) \cdot \boldsymbol{E}}{c} \\ &= \frac{(\boldsymbol{\nu} \times \boldsymbol{m}) \cdot \boldsymbol{E}}{c} - \frac{(\boldsymbol{\nu} \times \boldsymbol{m}_0) \cdot \boldsymbol{E}}{c} \\ &= \frac{(\boldsymbol{\nu} \times (\boldsymbol{m} - \boldsymbol{m}_0)) \cdot \boldsymbol{E}}{c} \\ &= \frac{(\gamma - 1)}{\gamma v^2} (\boldsymbol{m}_0 \cdot \boldsymbol{\nu}) \frac{(\boldsymbol{\nu} \times \boldsymbol{\nu}) \cdot \boldsymbol{E}}{c} = 0, \end{aligned}$$

where we have used the vector identity $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$, as well as the transformation (5) at $\boldsymbol{p}_0 = 0$.

Here we should like to point out that in the analysis performed just above, the assumption about the independence of electric and magnetic fields on time is important: otherwise, an electric field varying with time induces a magnetic field, and vice versa, which makes the situation more complicated.

Finally, it is worth emplasizing that in the ultra-relativistic case, the energy (44) essentially depends on the mutual orientation of vectors $\boldsymbol{p}_0$, $\boldsymbol{E}$, $\boldsymbol{\nu}$ and $\boldsymbol{m}_0$, $\boldsymbol{B}$, $\boldsymbol{\nu}$.

First of all, we consider the fraction of energy (44), which includes the electric dipole moment, *i.e.*

$$E_p = -\frac{(\boldsymbol{p}_0 \cdot \boldsymbol{E})}{\gamma^2} - \frac{(\boldsymbol{p}_0 \cdot \boldsymbol{\nu})(\gamma - 1)}{\gamma^2 v^2} (\boldsymbol{\nu} \cdot \boldsymbol{E}). \tag{50}$$

We see that in the particular case, where vectors $\boldsymbol{p}_0$, $\boldsymbol{E}$, $\boldsymbol{\nu}$ are collinear to each other, this energy is equal to

$$(E_p)_{//} = -\frac{(\boldsymbol{p}_0 \cdot \boldsymbol{E})}{\gamma}\,. \tag{51a}$$

Furthermore, in the case, where one of the vectors ($\boldsymbol{p}_0$ or $\boldsymbol{E}$) is orthogonal to $\boldsymbol{\nu}$, the energy becomes

$$(E_p)_{\perp} = -\frac{(\boldsymbol{p}_0 \cdot \boldsymbol{E})}{\gamma^2}\,. \tag{51b}$$

The difference between eqs. (51a) and (51b) is actually crucial at $\gamma \gg 1$, and it should be taken into account in practical applications.

From the physical viewpoint, the appearance of factor $1/\gamma$ in eq. (51a) is explained by the contraction of the length of the electric dipole along the velocity $\boldsymbol{\nu}$, which induces the proportional decrease of its energy. The additional suppression of the energy (51b) $1/\gamma$ times in comparison with eq. (51a) in both possible cases ($\boldsymbol{p}_0 \perp \boldsymbol{\nu}$, and $\boldsymbol{E}$ is not collinear to $\boldsymbol{\nu}$; $\boldsymbol{E} \perp \boldsymbol{\nu}$, and $\boldsymbol{p}_0$ is not collinear to $\boldsymbol{\nu}$) is caused by the interaction of the magnetic dipole moment $\frac{\boldsymbol{p}_0 \times \boldsymbol{\nu}}{c}$ with the electric field (*i.e.* the contribution of the hidden momentum of the magnetic dipole, emerging due to the motion of the electric dipole), which thus reduces the value of the electromagnetic energy of the entire bunch of charges in comparison with energy (51a). For more explanation of this effect see ref. [14].

The energy of a magnetic dipole in a magnetic field is defined by the expression

$$E_m = -\frac{(\boldsymbol{m}_0 \cdot \boldsymbol{B})}{\gamma^2} - \frac{(\boldsymbol{m}_0 \cdot \boldsymbol{\nu})(\gamma - 1)}{\gamma^2 v^2}(\boldsymbol{\nu} \cdot \boldsymbol{B}), \tag{52}$$

which is symmetric to eq. (50) with respect to the replacements $\boldsymbol{p}_0 \to \boldsymbol{m}_0$, $\boldsymbol{E} \to \boldsymbol{B}$. Correspondingly, all the comments made above on eq. (50) remain true with respect to eq. (52), too. In particular, the energy of the magnetic dipole is equal to

$$(E_m)_{//} = -\frac{(\boldsymbol{m}_0 \cdot \boldsymbol{B})}{\gamma} \tag{53a}$$

in the case $\boldsymbol{m}_0 // \boldsymbol{B} // \boldsymbol{\nu}$, and

$$(E_m)_{\perp} = -\frac{(\boldsymbol{m}_0 \cdot \boldsymbol{B})}{\gamma^2}\,, \tag{53b}$$

when one of the vectors ($\boldsymbol{m}_0$ or $\boldsymbol{B}$) is orthogonal to $\boldsymbol{\nu}$. Again, the difference between eqs. (53a) and (53b) becomes crucial at $\gamma \gg 1$ (the ultrarelativiastic case), and this circumstance should be taken into account in practical applications, dealing with the relativistic motion of magnetic dipoles in an electromagnetic fields.

From the physical viewpoint, the appearance of factor $1/\gamma$ in eq. (53a) (when vectors $\boldsymbol{m}_0$ and $\boldsymbol{\nu}$ are collinear to each other) is explained by the time dilation effect for the moving magnetic dipole, which decreases the current of the dipole and its magnetic dipole moment by $1/\gamma$ times. The additional suppression of the energy (53b) $1/\gamma$ times in comparison with eq. (53a) in the case $\boldsymbol{m}_0 \perp \boldsymbol{\nu}$ is caused by the contraction of the area of the dipole along the direction of $\boldsymbol{\nu}$. In another case of validity of eq. (53b) ($\boldsymbol{B} \perp \boldsymbol{\nu}$, and $\boldsymbol{m}_0$ not collinear to $\boldsymbol{\nu}$) the mechanism of suppression of energy is more complicated and is skipped in this paper. We only mention that it combines the effects of relativistic contraction of the area of a magnetic dipole, as well as the interaction of the latent momentum of the electric dipole (emerging due to the motion of the magnetic dipole) with the magnetic field $\boldsymbol{B}$.

3 Conclusion

In this paper we collected altogrther the relativistic expressions for the force, torque and energy of a small electric/magnetic dipole in an electromagnetic field, which we recently obtained, and analyzed a number of subtle effects, related to the relativistic motion of dipoles.

In particular, we presented the solution of the paradox by Mansuripur with the explicit explanation of its origin.

We have shown that the obtained expressions for the force and torque on a moving dipole, along with the relativistic transformation of the electric/magnetic dipole moment, do include all of the known relativistic effects (in particular, the Thomas-Wigner rotation in successive space-time transformations) and provide adequate results from the relativistic point of view.

Finally, we have shown that the energy of the ultrarelativistic electric/magnetic dipole essentially depends on the mutual orientation of velocity, electric (magnetic) dipole moment and electric (magnetic) field.

References

1. V. Bargmann, L. Michel, V.L. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).
2. L.H. Thomas, Nature (London) **117**, 514 (1926).
3. J.D. Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, New York, 1998).
4. J.D. Jackson, Am. J. Phys. **72**, 1484 (2004).
5. W. Shockley, R.P. James, Phys. Rev. Lett. **18**, 876 (1967).
6. A.O. Barut, *Electrodynamics and classical theory of fields & particles* (Dover, New York, 1980).
7. V. Namias, Am. J. Phys. **57**, 171 (1989).
8. G.P. Fischer, Am. J. Phys. **39**, 1528 (1971).
9. J.E. Verstein, Eur. J. Phys. **18**, 113 (1997).

10. M. Mansuripur, Phys. Rev. Lett. **108**, 193901 (2012).
11. A.L. Kholmetskii, O.V. Missevitch, T. Yarman, Eur. J. Phys. **32**, 873 (2011).
12. A.L. Kholmetskii, O.V. Missevitch, T. Yarman, Eur. J. Phys. **33**, L7 (2012).
13. A.L. Kholmetskii, O.V. Missevitch, T. Yarman, Prog. Electromagn. Res. B **45**, 83 (2012).
14. A.L. Kholmetskii, O.V. Missevitch, T. Yarman, Can. J. Phys. **9**, 576 (2013).
15. W.K.H. Panofsky, M. Phillips, *Classical Electricity and Magnetism*, second edition (Addison-Wesley, Reading, MA, 1969).
16. A.L. Kholmetskii, O.V. Missevitch, T. Yarman, Prog. Electromagn. Res. B **47**, 263 (2013).
17. Y. Aharonov, P. Pearle, L. Vaidman, Phys. Rev. A **37**, 4052 (1988).
18. L. Vaidman, Am. J. Phys. **58**, 978 (1990).
19. V. Hnizdo, Eur. J. Phys. **33**, L7 (2012).
20. A. Einstein, J. Laub, Ann. Phys. (Leipzig) **331**, 532 (1908).
21. A. Einstein, J. Laub, Ann. Phys. (Leipzig) **331**, 541 (1908).
22. D.A. Vanzella, Phys. Rev. Lett. **110**, 089401 (2013).
23. S.M. Barnett, Phys. Rev. Lett. **110**, 089402 (2013).
24. P.L. Saldanha, Phys. Rev. Lett. **110**, 089403 (2013).
25. M. Khorrami, Phys. Rev. Lett. **110**, 089404 (2013).
26. C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1973).
27. V. Hnizdo, Magn. Electr. Sep. **3**, 259 (1992).